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Boundary value problem with integral condition for a Blasius type equation*

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Abstract. The steady motion in the boundary layer along a thin flat plate, which is immersed at zero incidence in a uniform stream with constant velocity, can be described in terms of the solution of the differential equation $x''' = -xx''$, which satisfies the boundary conditions $x(0) = x'(0) = 0$, $x'(\infty) = 1$. The author investigates the generalized boundary value problem consisting of the nonlinear third-order differential equation $x''' = -t^r x|x|^{q-1}x''$ subject to the integral boundary conditions $x(0) = x'(0) = 0$, $x'(\infty) = \lambda \int_0^\xi x(s)\,\mathrm{d}s$, where $0 < \xi < +\infty$ is a fixed number and $\lambda > 0$ is a parameter. Results on the existence and uniqueness of solutions to boundary value problem are established. An illustrative example is provided.

Keywords: boundary layer, Blasius equation, integral boundary conditions, existence and uniqueness of solutions.

1 Introduction

When an incompressible fluid flows close to solid boundaries, the Navier–Stokes equations can be reduced to the Prandtl boundary layer equations [6]

$$\frac{\partial u}{\partial \mathsf{x}} + \frac{\partial v}{\partial \mathsf{y}} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial \mathsf{x}} + v\frac{\partial u}{\partial \mathsf{y}} = \nu\frac{\partial^2 u}{\partial \mathsf{y}^2} + U\frac{\mathrm{d}U}{\mathrm{d}\mathsf{x}}, \tag{2}$$

where $U(\mathsf{x})$ is the free stream velocity, u and v are the velocity components in x and y directions, respectively, and ν is the kinematic viscosity. The appropriate boundary conditions are

$$u = v = 0 \quad \text{when } \mathsf{y} = 0, \qquad u \to U(\mathsf{x}) \quad \text{as } \mathsf{y} \to \infty.$$

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In the particular case of the steady two-dimensional incompressible boundary layer flow along a thin flat plate, the following similarity transformation is possible:

$$\eta = \mathsf{y}\sqrt{\frac{U}{\nu \mathsf{x}}}, \qquad f(\eta) = \frac{\psi}{\sqrt{\nu \mathsf{x} U}},$$

which leads equations (1) and (2) to the well known Blasius equation [1, 2]

$$\frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0 \tag{3}$$

with boundary conditions

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1. \tag{4}$$

Note that the Blasius equation (3) is a special case of the Falkner–Skan equation [2, 3]

$$\frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} + \frac{1}{2} f \frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + \beta\left[1 - \left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^2\right] = 0,$$

which describes the same phenomena, when the flat plate turns to a wedge of angle $\pi\beta$ ($0 \leqslant \beta < 2$).

The purpose of this paper is to study the existence and uniqueness of solutions to the boundary value problem consisting of the nonlinear third-order differential equation

$$x''' = -t^r x|x|^{q-1} x'' \tag{5}$$

subject to the integral boundary conditions

$$x(0) = x'(0) = 0, \qquad x'(\infty) = \lambda \int_0^{\xi} x(s)\,\mathrm{d}s, \tag{6}$$

where $r \geqslant 0$, $q \geqslant 1$, $0 < \xi < +\infty$ is a fixed number and $\lambda > 0$ is a parameter. If $r = 0$ and $q = 1$, then (5) coincides with the Blasius equation (3).

Interest in nonlocal boundary value problems for differential equations involving integral boundary conditions is due to the fact that they often appear in physics and in various branches of applied mathematics.

A survey of recent results on the existence and uniqueness of solutions of nonlocal boundary value problems was presented by Ntouyas [4]. The second-order Sturm–Liouville problem with one classical and the other nonlocal two-point or integral boundary condition was considered by Pečiulytė, Štikonienė and Štikonas in [5]. A survey of recent results on the Green's functions and on spectrum for stationary problems with nonlocal boundary conditions was provided by Štikonas in [9]. The second-order boundary value problem with a damping term subject to the integral conditions was considered by Sergejeva in [7]. The author has established the solvability results based on the knowledge of the Fučík type spectrum for the auxiliary problem.

The paper is organized as follows. In Section 2, we consider the properties of solutions of initial value problem (5), (7). Section 3 is devoted to the dependence of solutions on initial data. In Section 4, we establish results on the existence and uniqueness of solutions to boundary value problem (5), (6). Also one example is given to illustrate the results. In order the paper to be self-contained, we provide some auxiliary results, published in [8].

2 Properties of solutions

We denote by $x(t,\gamma)$ the solution of the auxiliary initial value problem for equation (5) with initial data

$$x(0) = x'(0) = 0, \qquad x''(0) = \gamma. \tag{7}$$

Let $[0, t_\gamma)$, $t_\gamma \leqslant \infty$ be the interval of existence of $x(t,\gamma)$.

Proposition 1. (*See* [8, *Prop.* 3].) *If $x(t,\gamma)$ is a solution of initial value problem* (5), (7) *with $\gamma \neq 0$, then $x''(t,\gamma) \neq 0$ for $0 < t < t_\gamma$.*

Corollary 1. (*See* [8, *Cor.* 4].) *Consider a solution $x(t,\gamma)$ of initial value problem* (5), (7). *If $\gamma > 0$, then $x'(t,\gamma)$ and $x(t,\gamma)$ are positive and increasing functions for $0 < t < t_\gamma$. Moreover, $x''(t,\gamma)$ is a decreasing function for $0 < t < t_\gamma$. If $\gamma < 0$, then $x'(t,\gamma)$ and $x(t,\gamma)$ are negative and decreasing functions for $0 < t < t_\gamma$.*

Proposition 2. (*See* [8, *Prop. 5*].) *Consider a solution $x(t)$ of* (5). *If one of the functions $x(t)$, $x'(t)$ or $x''(t)$ is unbounded as $t \to t^* < +\infty$, then the others also are unbounded.*

Proposition 3. (*See* [8, *Prop.* 6].) *If $x(t,\gamma)$ is a solution of initial value problem* (5), (7) *with $\gamma > 0$, then $x(t,\gamma)$ is defined for all $t \geqslant 0$* ($t_\gamma = \infty$).

Proposition 4. (*See* [8, *Prop.* 7].) *Consider a solution $x(t,\gamma)$ of initial value problem* (5), (7). *If $\gamma > 0$, then there exists a positive constant k such that $\lim_{t\to+\infty} x'(t,\gamma) = k$.*

Remark. Every solution $x(t,\gamma)$ of initial value problem (5), (7) with $\gamma > 0$ has a slant asymptote $kt - b$, where $k > 0$, $b > 0$ are constants dependent on γ.

Proposition 5. *Consider a solution $x(t,\gamma)$ of initial value problem* (5), (7) *with $\gamma > 0$. If a line $kt - b$ is a slant asymptote for $x(t,\gamma)$, then, for $t > 0$,*

$$kt - b < x(t,\gamma) < kt.$$

Proof. Since $kt - b$ is a slant asymptote for $x(t,\gamma)$ and $x(t,\gamma)$, $x'(t,\gamma)$ $x''(t,\gamma)$ are positive functions for $t > 0$, then $kt - b < x(t,\gamma)$.

Next, let us prove that $x(t,\gamma) < kt$. Since $x'(t,\gamma)$ is an increasing and bounded function for $t > 0$ (see Corollary 1 and Proposition 4), then $x'(t,\gamma) < k$ for $t > 0$. Therefore, $\int_0^t x'(s,\gamma)\,\mathrm{d}s < \int_0^t k\,\mathrm{d}s$ or $x(t,\gamma) < kt$. □

3 Dependence on initial data

Proposition 6. (*See* [8, *Prop.* 8].) *If $x(t)$ is a solution of* (5)*, then the function*

$$y(t) = B^{(1+r)/q} x(Bt),$$

where $B > 0$ is an arbitrary constant, is also a solution of (5)*.*

Proposition 7. (*See* [8, *Prop.* 9].) *If $x(t,\gamma_0)$ is a solution of* (5) *with initial data*

$$x(0,\gamma_0) = x'(0,\gamma_0) = 0, \qquad x''(0,\gamma_0) = \gamma_0 \neq 0,$$

then every solution of equation (5)*, which has a double zero at $t = 0$ and the second derivative γ at $t = 0$ of the same sign as γ_0 ($\gamma\gamma_0 > 0$), can be expressed via solution $x(t,\gamma_0)$ as*

$$x(t,\gamma) = \left(\frac{\gamma}{\gamma_0}\right)^{(1+r)/(1+r+2q)} x\left(\left(\frac{\gamma}{\gamma_0}\right)^{q/(1+r+2q)} t, \gamma_0\right). \tag{8}$$

Proposition 8. *Consider a solution $x(t,\gamma_0)$ of initial value problem* (5)*,* (7) *with $\gamma_0 > 0$ fixed. If a line $k_0 t - b_0$ is a slant asymptote for $x(t,\gamma_0)$, then the line*

$$k_0 \left(\frac{\gamma}{\gamma_0}\right)^{(1+r+q)/(1+r+2q)} t - b_0 \left(\frac{\gamma}{\gamma_0}\right)^{(1+r)/(1+r+2q)}$$

is a slant asymptote for $x(t,\gamma)$ for every $\gamma > 0$.

Proof. Let $kt - b$ be a slant asymptote for $x(t,\gamma)$. Therefore,

$$\begin{aligned}
k &= \lim_{t\to+\infty} x'(t,\gamma) \\
&= \lim_{t\to+\infty} \left(\left(\frac{\gamma}{\gamma_0}\right)^{(1+r+q)/1+r+2q)} x'\left(\left(\frac{\gamma}{\gamma_0}\right)^{q/(1+r+2q)} t, \gamma_0\right)\right) \\
&= \left(\frac{\gamma}{\gamma_0}\right)^{(1+r+q)/(1+r+2q)} k_0
\end{aligned}$$

and

$$\begin{aligned}
b &= \lim_{t\to+\infty} \bigl(x(t,\gamma) - kt\bigr) \\
&= \lim_{t\to+\infty} \left(\left(\frac{\gamma}{\gamma_0}\right)^{(1+r)/(1+r+2q)} x\left(\left(\frac{\gamma}{\gamma_0}\right)^{q/(1+r+2q)} t, \gamma_0\right) \right. \\
&\quad \left. - k_0 \left(\frac{\gamma}{\gamma_0}\right)^{(1+r+q)/(1+r+2q)} t\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\gamma}{\gamma_0}\right)^{(1+r)/(1+r+2q)} \lim_{t\to+\infty} \left(x\left(\left(\frac{\gamma}{\gamma_0}\right)^{q/(1+r+2q)} t, \gamma_0\right) \right. \\
&\quad \left. - k_0 \left(\frac{\gamma}{\gamma_0}\right)^{q/(1+r+2q)} t \right) \\
&= \left(\frac{\gamma}{\gamma_0}\right)^{(1+r)/(1+r+2q)} b_0.
\end{aligned}
$$

Hence the proof. □

Corollary 2. *Consider a solution $x(t,\gamma_0)$ of initial value problem* (5), (7) *with $\gamma_0 > 0$. If a line $k_0 t - b_0$ is a slant asymptote for $x(t,\gamma_0)$, then, for every $\gamma > 0$ and $t > 0$,*

$$
\begin{aligned}
&k_0 \left(\frac{\gamma}{\gamma_0}\right)^{(1+r+q)/(1+r+2q)} t - b_0 \left(\frac{\gamma}{\gamma_0}\right)^{(1+r)/(1+r+2q)} \\
&\quad < x(t,\gamma) < k_0 \left(\frac{\gamma}{\gamma_0}\right)^{(1+r+q)/(1+r+2q)} t.
\end{aligned}
$$

Proof. The proof follows from Propositions 5 and 8. □

4 Existence and uniqueness result

Theorem 1. *If $\lambda\xi^2 \leqslant 2$, then boundary value problem* (5), (6) *is not solvable. If $\lambda\xi^2 > 2$, then boundary value problem* (5), (6) *has a unique solution.*

Remark. Since the classical Blasius equation (3) is a special case of equation (5), then Theorem 1 is valid for boundary value problem (3), (6).

Proof of Theorem 1. Let $x(t,\gamma_0)$ be a solution of initial value problem (5), (7) with $\gamma_0 > 0$ fixed, then every solution $x(t,\gamma)$ of initial value problem (5), (7) with $\gamma > 0$ can be expressed as (8) (see Proposition 7). Boundary value problem (5), (6) is solvable iff there exists $\gamma = \gamma_*$ such that $x'(\infty,\gamma_*) = \lambda X(\xi,\gamma_*)$, where $X(t,\gamma) = \int_0^t x(s,\gamma)\,\mathrm{d}s$, or

$$
k_0 \left(\frac{\gamma_*}{\gamma_0}\right)^{(1+r+q)/(1+r+2q)} = \left(\frac{\gamma_*}{\gamma_0}\right)^{(1+r-q)/(1+r+2q)} \lambda X\left(\left(\frac{\gamma_*}{\gamma_0}\right)^{q/(1+r+2q)} \xi, \gamma_0\right).
$$

By Corollary 2,

$$
x(t,\gamma) < k_0 \left(\frac{\gamma}{\gamma_0}\right)^{(1+r+q)/(1+r+2q)} t.
$$

Integrating from 0 to t, we obtain

$$
X(t,\gamma) < \frac{k_0}{2} \left(\frac{\gamma}{\gamma_0}\right)^{(1+r+q)/(1+r+2q)} t^2.
$$

Let $\lambda\xi^2 \leqslant 2$, then

$$\begin{aligned}\lambda X(t,\gamma) &= \left(\frac{\gamma}{\gamma_0}\right)^{(1+r-q)/(1+r+2q)} \lambda X\left(\left(\frac{\gamma}{\gamma_0}\right)^{q/(1+r+2q)} \xi, \gamma_0\right) \\ &< \lambda \frac{k_0}{2}\left(\frac{\gamma}{\gamma_0}\right)^{(1+r+q)/(1+r+2q)} \xi^2 \leqslant k_0 \left(\frac{\gamma}{\gamma_0}\right)^{(1+r+q)/(1+r+2q)}.\end{aligned}$$

Therefore, if $\lambda\xi^2 \leqslant 2$, then boundary value problem (5), (6) has no solutions.

Now, assume that $\lambda\xi^2 > 2$. Consider

$$\begin{aligned}&\lambda X(\xi,\gamma) - x'(\infty,\gamma) \\ &\quad = \left(\frac{\gamma}{\gamma_0}\right)^{(1+r-q)/(1+r+2q)} \lambda X\left(\left(\frac{\gamma}{\gamma_0}\right)^{q/(1+r+2q)} \xi, \gamma_0\right) \\ &\qquad - k_0\left(\frac{\gamma}{\gamma_0}\right)^{(1+r+q)/(1+r+2q)} \\ &\quad = \left(\frac{\gamma}{\gamma_0}\right)^{(1+r-q)/(1+r+2q)} \left(\lambda X\left(\left(\frac{\gamma}{\gamma_0}\right)^{q/(1+r+2q)} \xi, \gamma_0\right)\right. \\ &\qquad \left. - k_0\left(\frac{\gamma}{\gamma_0}\right)^{2q/(1+r+2q)}\right) \\ &\quad = \left|\left(\frac{\gamma}{\gamma_0}\right)^{2q/(1+r+2q)} = \left(\frac{s}{\xi}\right)^2\right| \\ &\quad = \left(\frac{\gamma}{\gamma_0}\right)^{(1+r-q)/(1+r+2q)} \left(\lambda X\left(s,\gamma_0\right) - \frac{k_0}{\xi^2} s^2\right).\end{aligned}$$

Consider the auxiliary function $g(s) = \lambda X(s,\gamma_0) - (k_0/\xi^2)s^2$, $g'(s) = \lambda x(s,\gamma_0) - (2k_0/\xi^2)s$. Thus, there exists a unique $s = s_1 > 0$ such that $g'(s_1) = 0$ and $g'(s) < 0$ for $0 < s < s_1$ and $g'(s) > 0$ for $s > s_1$. Consider $g''(s) = \lambda x'(s,\gamma_0) - (2k_0/\xi^2)$. Thus, there exists a unique $s = s_2 > 0$ such that $g''(s_2) = 0$ and $g''(s) < 0$ for $0 < s < s_2$ and $g''(s) > 0$ for $s > s_2$. Therefore, there exists a unique $s = s_*$ such that $g(s_*) = 0$ or $\lambda X(s_*,\gamma_0) - (k_0/\xi^2)s_*^2 = 0$.

Therefore, there exists a unique $\gamma = \gamma_*$ such that $x'(\infty,\gamma_*) = \lambda X(\xi,\gamma_*)$ or boundary value problem (5), (6) is uniquely solvable for $\lambda\xi^2 > 2$. □

Example 1. Consider the problem

$$\begin{aligned}&x''' = -t^2 x^3 x'', \\ &x(0) = x'(0) = 0, \qquad x'(\infty) = 3\int_0^1 x(s)\,\mathrm{d}s.\end{aligned} \tag{9}$$

Since $\lambda\xi^2 = 3 > 2$, then boundary value problem (9) has a unique solution, which is presented in Fig. 1. Numerically we can obtain that $x'(\infty) = 3\int_0^1 x(s)\,\mathrm{d}s \approx 10.6 > x'(1)$.

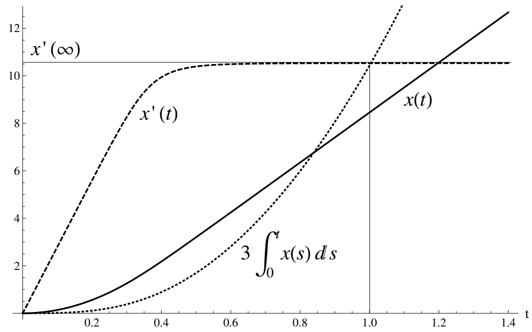

Figure 1. Functions $x(t)$ (solid), $x'(t)$ (dashed) and $3\int_0^t x(s)\,\mathrm{d}s$ (dotted).

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